

Bayesian Hypothesis Testing for Proportions

Antonio Nieto / Sonia Extremera / Javier Gómez



Introduction

- Tests on proportions

- Frequentist approach

- If $p\text{value} < \text{significance level} \rightarrow \text{Null hypothesis will be rejected}$

- Bayesian approach

- Probability under any hypotheses $\rightarrow$ Comparison to see what is the most plausible alternative

Both approaches can coexist and they should be used in the statistical interest

Bernouilli distribution

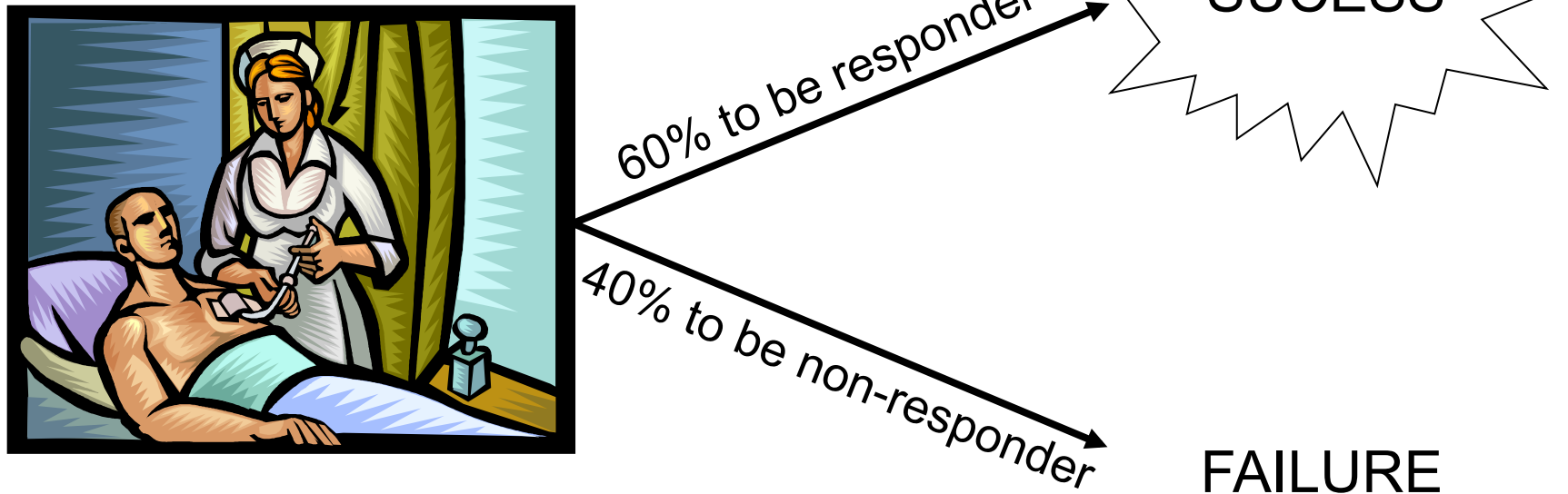
- The variable that records the patient's response follows a Bernouilli distribution
 - Discrete probability distribution, which takes value 1 “success” with probability “p” and 0 “failure” with probability “1-p”

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p = q & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = p \quad \text{Var}[x] = p \times q$$

Bernouilli

- Considering the probability to respond is 0.60



Binomial distribution

- Sum of “n” Bernouilli experiments

– Discrete probability distribution, which counts the sum of successes/failures out of ‘n’ independent samples

$$f(x) = \binom{n}{x} p^x \times q^{n-x} \quad \forall x = 0, 1 \dots n$$

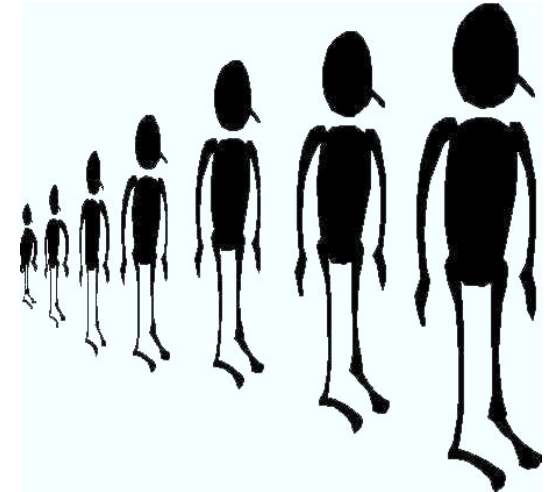
$$E[x] = n \times p \quad Var[x] = n \times p \times q$$

Binomial

Considering the probability to respond ($p=0.60$) in 10 patients then

$$E(x)=10 \times 0.6=6 \quad \text{Var}(x)=10 \times 0.6 \times 0.4=2.4$$

Exact confidence intervals, hypothesis tests can be calculated, binomial could be also approximated by the Normal distribution



Frequentist approach

A possible solution: Binomial distribution will be approximated with the Normal distribution and then taking a decision based on the pvalue associated to the Gauss curve

$$H_0 : p = p_0$$

$$H_1 : p \neq p_0$$

where p_0 is a pre - fixed value

$$\hat{p} = \frac{x}{n} \approx N\left(p_0, \frac{p_0 \times q_0}{n}\right) \Leftrightarrow \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \times q_0}{n}}} = Z \approx N(0,1)$$

Bayes' theorem (1763)

- It expresses the conditional probability of a random event A given B in terms of the conditional probability distribution of event B given A and the marginal probability of only A
- Let $\{A_1, A_2, \dots, A_n\}$ a set of mutually exclusive events, where the probability of each event is different from zero. Let B any event with known conditional probability $p(B|A_i)$. Then, the probability of $p(A_i|B)$ is given by the expression:

$$p(A_i | B) = \frac{p(B | A_i) \times p(A_i)}{p(B)}$$

where :

- $p(A_i)$ a priori probability
- $p(B | A_i)$ probability of B in A_i
- $p(A_i | B)$ posteriori probability

Bayes' in medicine

- **Sensitivity:** Probability of positive test when we know that the person suffers the disease
- **Specificity:** Probability of negative test when we know that the person does not suffer the disease

Probability of hypertension=0.2, sensitivity=91% specificity=98%

Probability to have hypertension if positive test is obtained

$$p = 0.91 \times 0.2 / (0.91 \times 0.2 + (1 - 0.98) \times 0.8) = 0.9192$$

Bayesian approach

- A priori distribution
- Sample distribution
- Posterior conjugate distribution

Beta distribution

- Continuous distribution in the interval (0,1)

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}; a > 0, b > 0; \Gamma(n) = (n-1)!$$

$$E[x] = \frac{a}{a+b} \quad Var[x] = \frac{ab}{(a+b+1)(a+b)^2}$$

- Posterior Beta (a,b) where $a = \sum x_i + \alpha$, $b = n - \sum x_i + \beta$

No 'a priori' information

- As initial assumption probability any value between zero and one $\longrightarrow$ Uniform $(0,1)=\text{Beta}(1,1)$
- Sample distribution Binomial (n,p)
- Posterior Beta (a,b) where $a=\sum x_i+1$, $b=n-\sum x_i+1$

Example 1

N=40, no prior information:

- H0: Proportion of responders is $\leq 40\%$
- H1: Proportion of responders is $> 60\%$

If 24 successes then posterior probability Beta (25,17)

H0	H1	X	N	Test	Prob. under H0	Prob. under H1
$p \leq 0.4$	$p > 0.6$	24	40	H1 is more probable than H0	0.005347226	0.48303

Prior distribution: Uniform (0,1)

Prior Knowledge

- Bayesian tests is enhanced when some information is available
 - Example the probability will fall [0.3-0.7]
 - In values relatively high of α and β , Beta~Normal then >95% of the probability $[m \pm 2s]$; where m =mean and s =standard deviation (s)
 - By means of a moment's method type
 - $m = \alpha / (\alpha + \beta)$; $s^2 = m(1-m) / (\alpha + \beta + 1)$
 - $\alpha = [m^2 (1-m) / s^2] - m$; $\beta = (\alpha - m\alpha) / m = [m (1-m)^2 / s^2] + m - 1$
- Sample distribution Binomial (n, p)
- Posterior Beta (a, b) where $a = \sum x_i + \alpha$, $b = n - \sum x_i + \beta$

Example 2

N=40, probability will fall [0.3-0.7] with a 95% probability:

- H0: Proportion of responders is $\leq 40\%$
- H1: Proportion of responders is $> 60\%$

If 24 successes then posterior probability Beta (36,28)

H0	H1	X	N	Test	Prob. under H0	Prob. under H1
$p \leq 0.4$	$p > 0.6$	24	40	H1 is more probable than H0	0.004406341	0.27539

Prior distribution: Beta (12,12)

SAS® macro

```
%MACRO Bayes_test (x=,n=,HO=,H1=,alpha=,beta=);

DATA bayes1;
length test $255.;
a=&x+&alpha;
b=&n-&x+&beta;
h0="p<="||left(trim(&HO));
h1="p>"||left(trim(&H1));
x=&x;
n=&n;

x1=probbeta(&HO,a,b);
x2=1-probbeta(&H1,a,b);

If x1>x2 then test='HO is more probable than H1';
else if x1<x2 then test='H1 is more probable than HO';
else if x1=x2 then test='Equally probable hypotheses';

run;

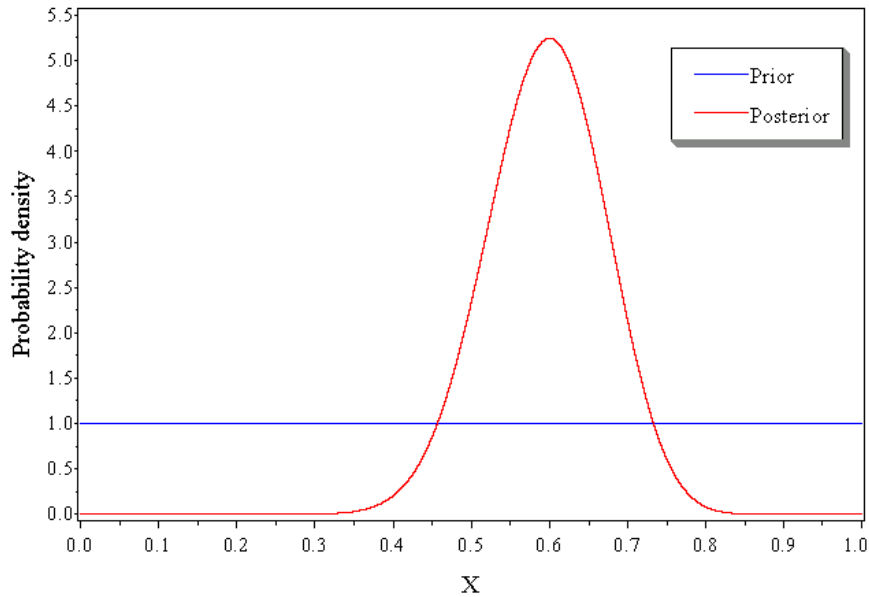
Proc print data=bayes1 noobs 1;
var h0 h1 x n test x1 x2;
label h0='HO' h1='H1' x='X' n='N' test='Test' x1='Prob. under HO' x2='Prob. under H1';
title "Bayes test of &x successes in &n samples";
footnote "Prior distribution: Beta (&alpha,&beta)";
run;
title;
footnote;

%MEND Bayes_test;

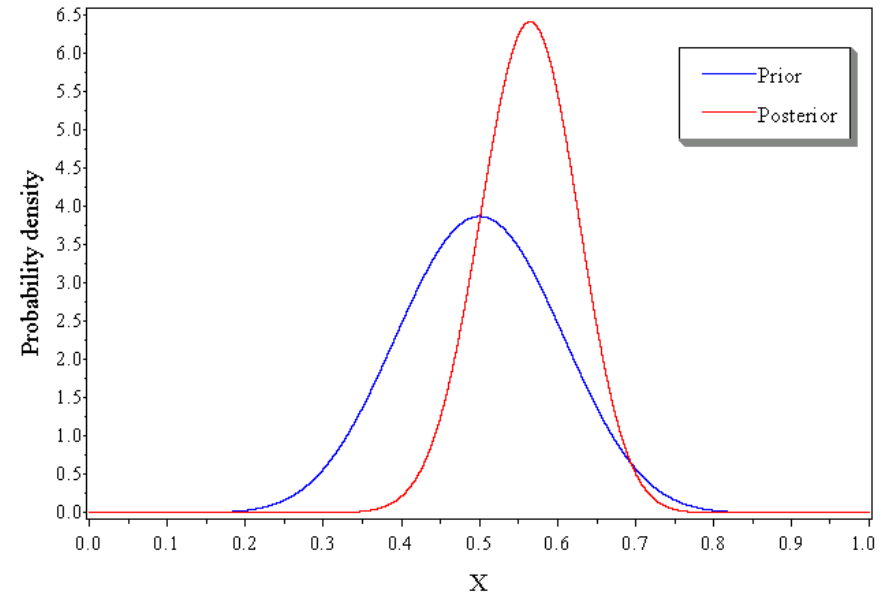
/*Example 1*/%Bayes_test (x=24,n=40,HO=0.40,H1=0.60,alpha=1,beta=1);
/*Example 2*/%Bayes_test (x=24,n=40,HO=0.40,H1=0.60,alpha=12,beta=12);
```

Beta distribution plots

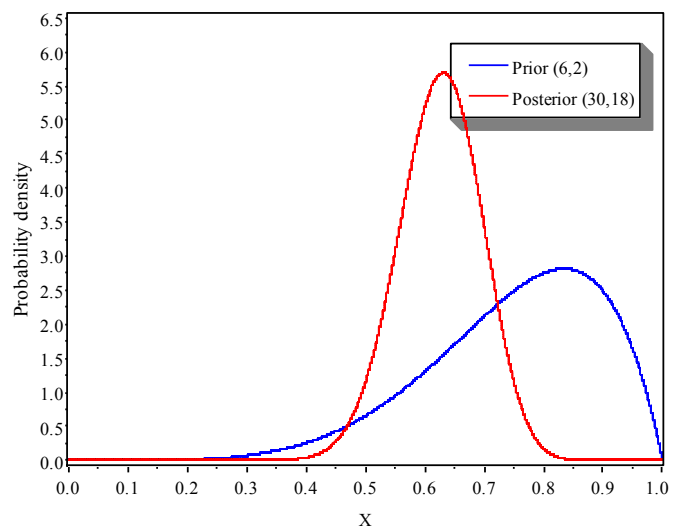
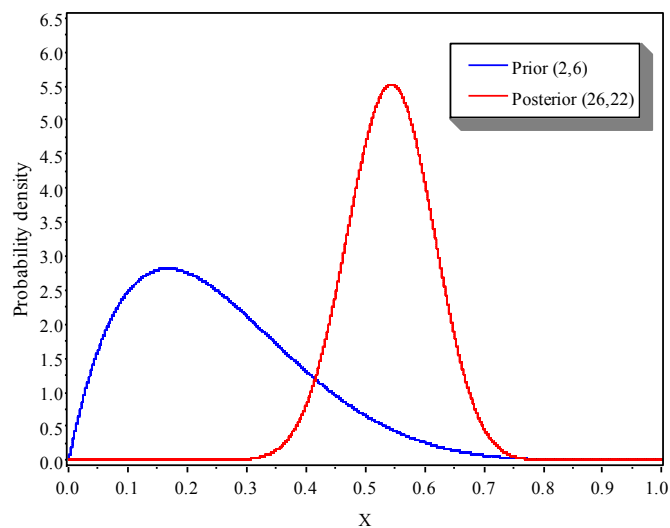
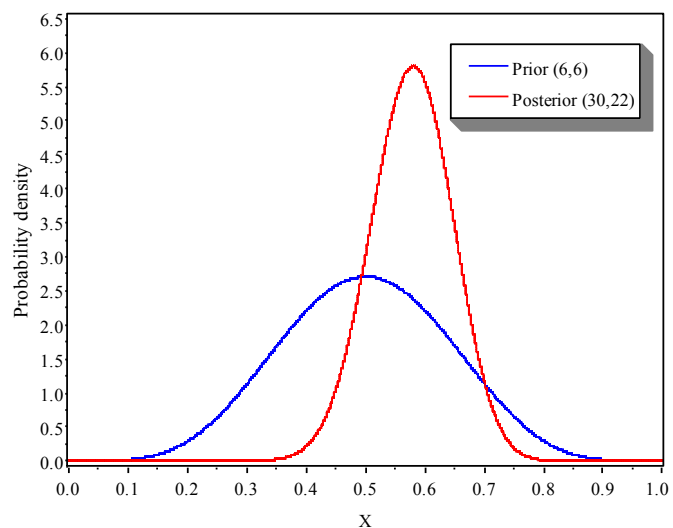
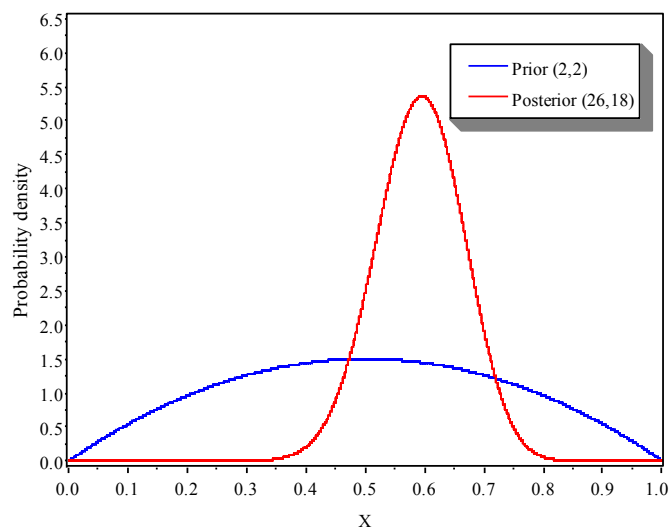
Example 1



Example 2



Example 2 (other prior)



Conclusion

- Bayesian tests are nowadays being increasingly used, especially in the context of adaptive designs
- Very important aspects are:
 - Good selection of the distributions
 - Clear definition of the "a priori" information collected
- A Bayesian approach has been presented to be included in the statistical armamentarium to test proportion hypotheses
 - It can be also extended to other endpoints and distributions

Questions

