

Introduction to numeric precision and representation issues. Why 4.8 minus 4.6 is not always equal to 0.2

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ABSTRACT

Numeric precision and representation issues are well-known topics in computer science. Non computer scientists sometimes are not aware of the problems that occur with these issues. This paper provides a brief introduction to problems based on numeric precision and the presentation of numbers on computerized systems.

Numbers are stored as binary numbers in computerized systems. However, not all floating-point numbers can be represented properly in the binary system. When calculating with numbers that cannot be stored exactly, the result may not be as expected. In our case 4.8 and 4.6 can not be stored exactly and therefore the result of 4.8 minus 4.6 will not be exactly 0.2.

The conversion of floating-point numbers to a storable IEEE-754 binary number is illustrated. Finally some SAS functions and options are introduced that may help to deal with the issue.

INTRODUCTION

As illustrated in figure 1 below, the value "0.200" in dataset WORK.A is not equal to the same value in dataset WORK.B. At this point it is important to know that this issue is SAS specific but rather a general issue in computer science.

The COMPARE Procedure
Comparison of WORK.A with WORK.B
(Method=EXACT)

All Variables Compared have Unequal Values

Variable	Type	Len	Ndif	MaxDif
B	NUM	8	1	167E-18

Value Comparison Results for Variables

Obs	Base B	Compare B	Diff.	% Diff
1	0.2000	0.2000	1.665E-16	8.327E-14

Figure 1: PROC COMPARE output

The problem is that decimal numbers are stored as binary numbers in computerized systems and thus have to be converted for storing. Often decimal numbers can only be represented with infinite binary numbers, but computerized systems are not able to store infinite numbers, which leads to loss of precision. When calculating with imprecise values the result must not necessarily be correct. This can cause problems when comparing values.

Of course we expect that a computerized system can handle a simple calculation like 4.8 - 4.6 and so when we create a dataset in SAS containing the result of this calculation, it seems to be correct (see figure 2).

```
data example1;
    result = 4.8 - 4.6;
    put result=;
    output;
run;
```

```
LOG:
RESULT=0.2
```

Figure 2: Calculation of 4.8 – 4.6

Now to be sure that the result is correct the value stored in dataset “example1” will be compared with the actual value of 0.2.

```
data example2;
    set example1;

    if result eq 0.2 then equal = "Y";
    else equal = "N";

    put equal=;
run;
```

```
LOG:
EQUAL=N
```

Figure 3: Comparison of the result with 0.2

The comparison (see figure 3) showed that the result of the calculation 4.8 - 4.6 is not equal to 0.2 on the system used for this test, even though SAS displays us the correct values in the dataset. It is important to note that SAS uses formatted or rounded values for representation.

To understand why a simple calculation like this goes wrong, it is important to understand how floating-point numbers are stored on computerized systems.

HOW FLOATING-POINT NUMBERS ARE STORED

SAS offers the functionality to display the binary and hexadecimal values of numbers. For better readability the hexadecimal values of our example are displayed (figure 4).

```
data example3;
    set example2;

    resulthex = put(result,hex16.);
    zeropointtwo = put(0.2,hex16.);

    put resulthex=;
    put zeropointtwo=;
run;
```

```
LOG:
RESULTHEX      = 3FC999999999999A0
ZEROPOINTTWO   = 3FC9999999999999A
```

Figure 4: Hexadecimal output

The output shows that the values are “almost” equal. They have differences in the last two digits of the hexadecimal values. This differences result from the calculation. The result of the subtraction is not exact because 4.8 and 4.6 could not be stored exactly.

The following sections illustrate how the numbers are stored in computerized systems. Only a general overview on how numbers are stored, is given since there are a lot of factors (e.g. the hardware used, the operating system, the programming language used etc.) that impact the precision of a system.

The IEEE Standard for Floating-Point Arithmetic (IEEE-754) is the most widely used standard for floating-point computation so that the following explanation will be based on this standard as far as possible.

CONVERSION TO THE BINARY SYSTEM

In our daily life we usually use the decimal system. In this system the numbers are represented in digits from 0 to 9 and the set value is represented to the power of the base 10.

As example we will use the number 225.75, representing a coefficient to the power of 10 (shown in figure 5).

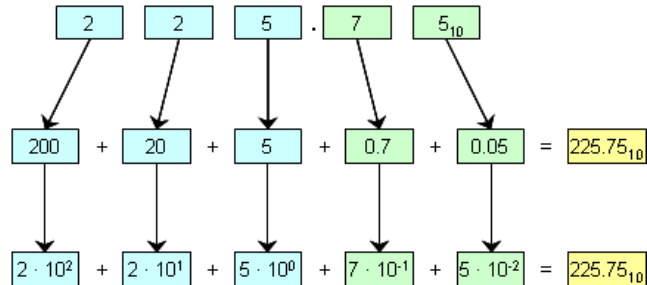


Figure 5: 225.75 in digits and in the set value to the power of the base 10

In computerized systems numbers are stored as binary numbers. The base for the binary system is 2 and numbers are represented in the digits 0 and 1.

The number 225.75 from the previous example converted to binary is shown in figure 6.

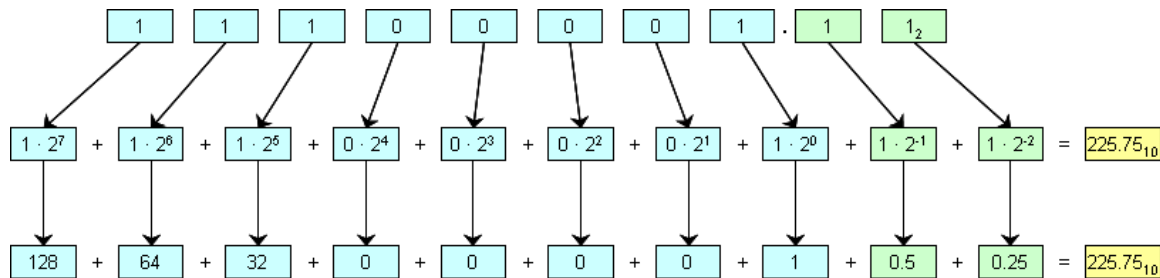


Figure 6: 225.75 as binary number

But how does a computerized system get from the decimal system to the binary system?

To convert the number 225.75 from the decimal system to the binary system the integer part and the decimal part are considered separately.

Firstly, the integer part of the number, "225", is converted into the binary system. To do this you have to take the following steps:

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1. Divide the number by the base (2) of the target number system. The remainder will represent a binary digit in the binary number. Use the integer part of the result for calculations in the following steps.
2. If the result of the division in step 1 is zero then you are finished.
3. If not use the integer from step 1 as the next number to calculate with and repeat steps 1 and 2 until the integer part of the result equals zero.
4. The first calculated binary digit represents the last digit in the binary number.

$225 : 2 = 112$ Remainder: 1 (Binary digit: 2^0)
 $112 : 2 = 56$ Remainder: 0 (Binary digit: 2^1)
 $56 : 2 = 28$ Remainder: 0 (Binary digit: 2^2)
 $28 : 2 = 14$ Remainder: 0 (Binary digit: 2^3)
 $14 : 2 = 7$ Remainder: 0 (Binary digit: 2^4)
 $7 : 2 = 3$ Remainder: 1 (Binary digit: 2^5)
 $3 : 2 = 1$ Remainder: 1 (Binary digit: 2^6)
 $1 : 2 = 0$ Remainder: 1 (Binary digit: 2^7)

Result:	1	1	1	0	0	0	0	1
Binary digit:	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Next, the decimal part of the number, "0.75", is converted into the binary system.

To do this you have to take the following steps:

1. Multiply the number by the base (2) of the target number system. The number before the decimal point will be the binary digit (starting from left to right).
2. The number after the decimal point will be the new number to calculate with.
3. Repeat steps 1 and 2 until you reach a number without a decimal point, a result that has already been produced, or the required precision.

$2 \cdot 0.75 = 1.5$ digit: 1
 $2 \cdot 0.5 = 1$ digit: 1

Result:	0	.	1	1
Binary digit:	2^0	.	2^{-1}	2^{-2}

Together with the previous converted integer part you will get complete binary number:

1	1	1	0	0	0	0	1	.	1	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	.	2^{-1}	2^{-2}

NORMALIZED FLOATING-POINT REPRESENTATION

Since computerized systems do not store decimal points the binary number ($11100001.11_2 = 225.75_{10}$) can not be stored like this. That is why numbers have to be converted to the normalized floating-point representation form.

Normally floating-point numbers consist of the integer part and the fractional part separated by a decimal point. The normalized floating-point representation form consists of the mantissa, the base of the number system used and an integer exponent. These three are illustrated in figure 7.

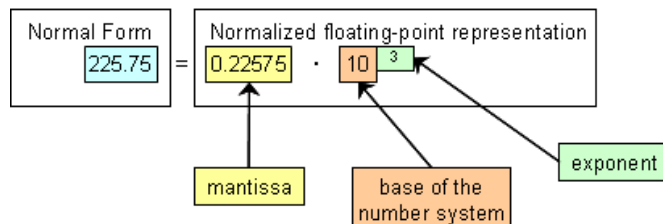


Figure 7: Normalized floating-point representation

The term "normalized" refers to the mantissa. It is always < 1 but ≥ 0.1 . Thus the only correct normalized representation form for 225.75 is $0.22575 \cdot 10^3$ and not $0.022575 \cdot 10^4$ or $2.2575 \cdot 10^2$.

Example: $225.75_{10} = 0.22575_{10} \cdot 10^3$

To bring the binary number from the previous examples into the normalized floating-point representation form we need to move the decimal point to the left and use the number of digits we moved the point as exponent to the basis.

$$11100001.11_2 = 0.1110000111_2 \cdot 2^8$$

Floating point numbers are basically stored like this in computerized systems. There are, however, several limitations. For example, due to the limited number of digits that can be stored numbers in computerized systems are limited to the value range of the exponent and the mantissa (for further explanations see the following sections).

FLOATING-POINT NUMBERS ACCORDING TO IEEE-754 CODING

The following explanation will give only a short overview to IEEE-754 coding.

According to IEEE-754 numbers are stored as shown in figure 8. Depending on the precision you are working with, you can store larger exponents and larger mantissas.

Sign of the mantissa	Biased exponent	Mantissa in the 1-plus-Form	Total number of bits
1 Bit 0 for positive 1 for negative	Single precision: 8 Bits Double precision: 11 Bits Quadruple precision: 15 Bits	23 Bits 52 Bits 112 Bits	32 Bits 64 Bits 128 Bits

Figure 8: Layout of floating-point numbers according to IEEE-754

The following steps are necessary to create a storable binary number according to IEEE-754:

1-PLUS-FORM

The so called 1-plus-form is generated by shifting the mantissa of the normalized floating-point representation form to the left by one digit. For our example this would look like:

$$0.1110000111_2 \rightarrow 1.110000111_2$$

The first digit must not be saved since it is always 1 due to the 1-plus-form. This fact is the advantage of the 1-plus-form in comparison to the normalized floating-point representation. One bit less is used to store the number, which means you have one additional bit available for storage. The mantissa to be saved will be: 110000111₂. In comparison to the normalized form the exponent is also different because the decimal point is moved one digit less in the 1-plus-form. In the 1-plus-form the exponent for this number is 7 while it is 8 in the normalized floating-point representation.

BIASED EXPONENT

The biased exponent makes it possible to store both positive and negative exponents without the need for an additional sign bit. Numbers smaller than the bias, are negative exponents and numbers greater than the bias are positive exponents.

The biased exponent is calculated as follows:

$$\text{Bias} + \text{original exponent (from 1-plus-form)} = \text{biased exponent}$$

The bias for the different number formats is:

Single precision:	127
Double precision:	1023
Quadruple precision:	16383

For our example of a double precision number this would mean:

$$1023_{10} + 7_{10} = 1030_{10} = 10000000110_2$$

FINALIZE

Now all the components are available to store the number. The number 225.75 as a double precision binary number would be stored as shown in the following picture. In case that a mantissa does not need all the bits available it will be filled with 0s at the end.

[illegible]

Figure 9: 225.75 as storable binary number according to IEEE-754

When storing negative numbers, the process is equal to positive number. The sign bit of the mantissa is set to 1 and the stored mantissa and biased exponent are equal to the positive number.

So the conversion of a decimal number to a storable binary number for computerized systems is processed according to the flowchart shown in figure 10.

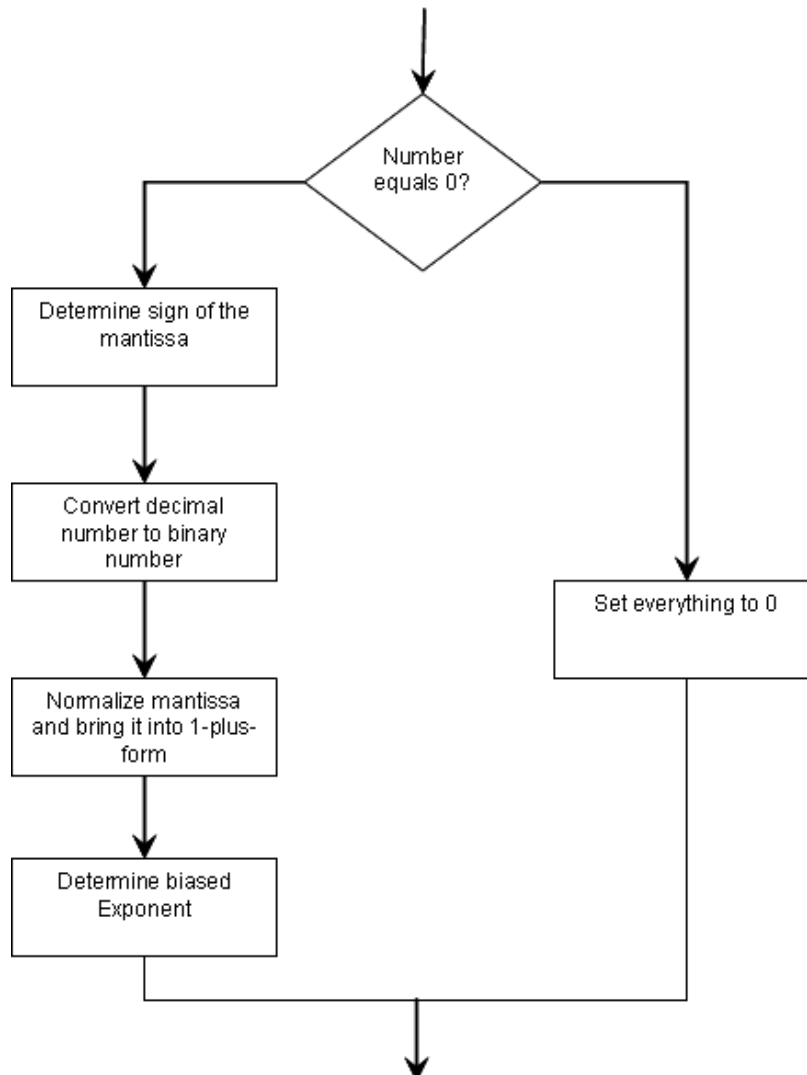


Figure 10: 225.75 in digits and in the set value to the power of the base 10

The conversion from binary back to decimal is done by logically following the flowchart backwards.

WHY 4.8 - 4.6 IS NOT ALWAYS EQUAL TO 0.2

After this example illustrating how numbers are converted and stored in computerized systems, it is still not clear why $4.8 - 4.6$ is not equal to 0.2 . In the following numbers 4.8 and 4.6 are converted to see what happens during the storing process.

The sign bit will be 0 for both numbers.

The integer part 4 is converted into the binary system (for both numbers):

$4 : 2 = 2$ Remainder: 0 (Binary digit: 2^0)
 $2 : 2 = 1$ Remainder: 0 (Binary digit: 2^1)
 $1 : 2 = 0$ Remainder: 1 (Binary digit: 2^2)

Result:	1	0	0
Binary digit:	2^2	2^1	2^0

The decimal part 0.8 is converted into the binary system:

$2 \cdot 0.8 = 1.6$ digit: 1
 $2 \cdot 0.6 = 1.2$ digit: 1
 $2 \cdot 0.2 = 0.4$ digit: 0
 $2 \cdot 0.4 = 0.8$ digit: 0
 $2 \cdot 0.8 = 1.6$ digit: 1 $\rightarrow$ the result 1.6 already appeared once and the calculation will repeat periodically until all available digits are used.

Result:	0	.	1	1	0	0	...
Binary digit:	2^0	.	2^{-1}	2^{-2}	2^{-3}	2^{-4}	...

The result 1.6 already appeared in the first step. From this point the digits will repeat periodically. This is the root of the problem: Numbers are stored with a finite length in computerized systems, but we would need an infinite length to represent the decimal number 4.8 as a binary number.

Figure 11 shows that the further you go to the right of decimal point the smaller the number will get.

Decimal value:	4	2	1	.	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	...	...	...
Set value to the power of 2:	2^2	2^1	2^0		2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	...	...	...
Binary digits:	1	0	0	.	1	1	0	0	1	1	0	0	1	1	...

Figure 11: Number line of the binary system

Only fractions with a power of two in the denominator are finite in the binary system. All other fractions are infinite. Since computerized systems do not save infinite numbers the value gets truncated which means a loss of precision. In the end, the number is not stored as 4.8 , it is stored as nearly as possible to 4.8 . In single precision systems the mantissa gets truncated after the 52^{nd} digit. Be aware that the mantissa according to the 1-plus-form also includes the values before the decimal point. That means there are less than 53 bits available for the fractional. For 4.8 this means:

$$100.1100110011001100110011001... = 1.00110011001100110011001... \cdot 2^2$$

The biased exponent is: $1023 + 2 = 1025_{10} = 10000000001_2$

When composing everything to a double precision number the result is as shown in figure 12:

Sign of the mantissa	Biased exponent	Mantissa in the 1-plus-Form
0	1 0 0 0 0 0 0 0 0 1	001100110011001100110011001100110011001100110011001100110011
1 Bit 0 for positive 1 for negative	Double precision: 11 Bits	52 Bits

Figure 12: 4.8 as storable binary number according to IEEE-754

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The actual stored number is 4.7999997... and not 4.8.

4.6 is stored as follows:

The sign bit is 0. The integer part has already been converted to binary (see above).

The decimal part 0.6 is converted into the binary system:

$2 \cdot 0.6 = 1.2$ digit: 1
 $2 \cdot 0.2 = 0.4$ digit: 0
 $2 \cdot 0.4 = 0.8$ digit: 0
 $2 \cdot 0.8 = 1.6$ digit: 1
 $2 \cdot 0.6 = 1.2$ digit: 1 $\rightarrow$ the result 1.2 already appeared once.

Result:	0	.	1	0	0	1	...
Binary digit:	2^0	.	2^{-1}	2^{-2}	2^{-3}	2^{-4}	...

Again the number is infinite in the binary system.

Double precision numbers have 52 bits for the mantissa in the 1-plus-form. So the 1-plus-form for the number 4.6 looks as follows:

$$100.1001100110011001100110011... = 1.00100110011001100110011... \cdot 2^2$$

The biased exponent is: $1023 + 2 = 1025_{10} = 10000000001_2$

When we compose everything to a double precision number the result is as shown in figure 13:

Sign of the mantissa	Biased exponent	Mantissa in the 1-plus-Form
0	1 0 0 0 0 0 0 0 0 1	001001100110011001100110011001100110011001100110011001100110
1 Bit 0 for positive 1 for negative	Double precision: 11 Bits	52 Bits

Figure 13: 4.6 as storable binary number according to IEEE-754

The actual stored number is 4.5999999... and not 4.6.

SAS displays these values as 4.8 and 4.6 because it uses rounding according to IEEE-754. This is one of the reasons why programmers often are not aware of this problem. However, when calculating with these values SAS and other systems use the binary numbers without rounding them. The system is therefore actually calculating "almost" 4.8 minus "almost" 4.6 and therefore the result will not be exactly 0.2. There will be a loss of precision when calculating with this kind of numbers.

The result of the subtraction above stored as binary is shown in figure 14:

Sign of the mantissa	Biased exponent	Mantissa in the 1-plus-Form
0	0 1 1 1 1 1 1 1 0 0	1001100110011001100110011001100110011001100110100000
1 Bit 0 for positive 1 for negative	Double precision: 11 Bits	52 Bits

Figure 14: Binary result of the calculation of 4.8 – 4.6

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While the actual stored value for 0.2 (the number as it is, not as a result of a calculation) is:

Sign of the mantissa	Biased exponent	Mantissa in the 1-plus-Form
0	0 1 1 1 1 1 1 1 0 0	10011001100110011001100110011001100110011010
1 Bit 0 for positive 1 for negative	Double precision: 11 Bits	52 Bits

Figure 15: 0.2 as storable binary number according to IEEE-754

Thus when comparing the mantissa it becomes apparent that the last 6 bits are not equal.

Mantissa of the result of calculation: 10011001100110011001100110011001100110011001100000

Mantissa of the actual value 0.2 : 10011001100110011001100110011001100110011001101010

This is the reason why, in the output of the following SAS code (figure 16), the numbers are not equal.

```
data example4;
  if 4.8 - 4.6 eq 0.2 then equal = "Y";
  else equal = "N";

  put equal=;
run;
```

LOG:

EQUAL=N

Figure 16: Comparison of the result with 0.2

It may happen that this example is equal on your system. This may be because your system is more precise. It may use more bits to store the mantissa and the exponent or it may round the values by default before comparing or representing them. The precision of a system depends on multiple factors like the hardware, the operating system or the programming language used.

CONCLUSION

It is possible to avoid this issue by simply using the round function whenever possible. The use of the round function in our SAS example is shown in figure 17.

```
data example5;
  if round(4.8 - 4.6, 0.1) eq 0.2 then equal = "Y";
  else equal = "N";

  put equal=;
run;
```

LOG:

EQUAL=Y

Figure 17: Comparison of the result with 0.2 using the round function

There are a lot of other helpful functions that may help to handle the problem. The following table shows a list of these functions.

Function	Syntax	Description
CEIL	CEIL(argument)	The CEIL function returns the smallest integer that is greater than or equal to the argument.
CEILZ	CEILZ(argument)	The CEIL function returns the smallest integer that is greater than or equal to the argument, using zero fuzzing.
FLOOR	FLOOR(argument)	The FLOOR function takes one numeric argument and returns the largest integer that is less than or equal to the argument.
FLOORZ	FLOORZ(argument)	The FLOOR function takes one numeric argument and returns the largest integer that is less than or equal to the argument, using zero fuzzing.
FUZZ	FUZZ(argument)	Returns the nearest integer if the argument is within 1E-12.

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INT	INT(argument)	Returns the integer value.
INTZ	INTZ(argument)	Returns the integer value, using zero fuzzing.
MOD	MOD(argument-1, argument-2)	Returns the remainder from the division of the first argument by the second argument.
MODZ	MODZ(argument-1, argument-2)	Returns the remainder from the division of the first argument by the second argument, using zero fuzzing.
ROUND	ROUND(argument <,rounding-unit>)	Rounds the first argument to the nearest multiple of the second argument.
ROUNDE	ROUNDE(argument <,rounding-unit>)	Rounds the first argument to the nearest multiple of the second argument, and returns an even multiple when the first argument is halfway between the two nearest multiples.
ROUNDZ	ROUNDZ(argument <,rounding-unit>)	Rounds the first argument to the nearest multiple of the second argument, with zero fuzzing.

By default the option “method=EXACT” is used and values are compared exactly in PROC COMPARE.

The option “method=ABSOLUTE” is helpful when comparing datasets from different systems with PROC COMPARE. This option specifies the method for judging the equality of numeric values. When using this method, values are unequal if $ABS(y-x) > CRITERION$. This means that the two compared values are subtracted and if the magnitude is greater than the specified value in the “CRITERION=” option these values are unequal. “CRITERION=” is an option of PROC COMPARE and specifies the border of equality for the option “method=ABSOLUTE”. By the default the option “CRITERION=” has the value 0.00001, as marked in figure 18.

The output for the example in the introduction would then look like in figure 18.

```

The COMPARE Procedure
Comparison of WORK.A with WORK.B
(Method=ABSOLUTE Criterion=0.00001)

Data Set Summary

Dataset          Created          Modified    NVar      NObs
WORK.A           05JUL11:10:47:49  05JUL11:10:47:49    1         1
WORK.B           05JUL11:10:47:49  05JUL11:10:47:49    1         1

Variables Summary

Number of Variables in Common: 1.

Observation Summary

Observation      Base    Compare
First Obs        1        1
Last Obs         1        1

Number of Observations in Common: 1.
Total Number of Observations Read from WORK.A: 1.
Total Number of Observations Read from WORK.B: 1.

Number of Observations with Some Compared Variables Unequal: 0.
Number of Observations with All Compared Variables Equal: 1.

Values Comparison Summary

Number of Variables Compared with All Observations Equal: 1.
Number of Variables Compared with Some Observations Unequal: 0.
Total Number of Values which Compare Unequal: 0.
Total Number of Values not EXACTLY Equal: 1.
Maximum Difference: 1.6653E-16.

```

Figure 18: PROC COMPARE output with the option “method=ABSOLUTE”

The last three lines of the PROC COMPARE output show that there are no unequal values. However, PROC COMPARE also shows that there are values that were not exactly equal and that the maximum difference is to be found 15 digits after decimal point.

Numeric precision and representation issues are well known and often forgotten (especially while programming). These Issues do not only occur in SAS.

Numeric precision problems should be taken into account during programming to avoid lengthy searches for errors.

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All output reported in this paper was generated by SAS 9.2 on Red Hat Enterprise Linux Release 5.

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